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IS "CREATIVE SUBJECT" OF BROUWER A DESIGNER? – AN ANALYSIS OF INTUITIONIST MATHEMATICS FROM THE VIEWPOINT OF C-K DESIGN THEORY

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ABSTRACT

The paper considers one of the main constructive mathematical theories - the Intuitionist Mathematics. Brouwer, the father of Intuitionist Mathematics, describes mathematics as the study of *mental mathematical constructions* realized by *a creative subject*. This perspective on mathematics represents many similarities with more conventional design processes. By analyzing Brouwer work and mathematical philosophy, we identify the reasoning process Brouwer explains as a conceptive reasoning process. We point out some interesting parallels and similarities between Brouwer's work and another theory of creativity as conceptive reasoning process coming from design research – namely, the C-K design theory. We argue that, even though the concept space of C-K theory is not explicitly represented, Intuitionist Mathematics authorizes, and even welcome, concepts and the reasoning process of the creative subject implicitly makes use of them. This, result combined with many parallels between the C-K theory and Intuitionist Mathematics, opens up interesting research perspectives.

Keywords: C-K design theory, Intuitionism, creative subject, design and mathematics

1 INTRODUCTION

Are mathematicians designers? More precisely, can we assimilate a mathematician's reasoning process to that of a designer? If design can be defined as the construction of a definition of an object allowing its construction (or proving its existence, for that matter), then, certainly, it becomes legitimate to suspect that mathematicians conduct a kind of design process in their everyday exercise: what is mathematics if not defining new (mathematical) objects with new and unprecedented properties? Arguably, such intriguing questions deserves more in depth analysis and consideration than straight yes or no answers, which, anyway, would not be possible considering the multiplicity of mathematical practices and outputs *and* the difficulty in capturing the act of design in one universal definition.

Nevertheless, the present study attempts to investigate the above question within the context of a particular approach to mathematics, namely, the Intuitionist Mathematics of L.E.J. Brouwer [1-5]. In Brouwer's view, mathematics is essentially a constructive activity and a mathematician is a *constructor* or a *creative subject*. The mathematical activity of an idealized mathematician, the so-called creative subject, is an exercise performed through out time in such a way that any new mathematical object can only be constructed with whatever previous entities constructed thus far. However, this does not reduce mathematics to a mere combination determined beforehand and existing as a totality. The creative subject should construct the mathematical objects and he may do so with *free choices* – which give the mathematician the means to conceive new and unprecedented mathematical entities any time.

Brouwer defended that such mathematical constructions are not independent of the mind that created them and they do not correspond, at least, not necessarily, to any outside reality. This subjectivist conception of mathematics is somewhat surprising, if judged from the perspective of older and more fundamentally rooted mathematical philosophies, such as the Platonism or Logicism. We claim, throughout the paper, that the seeming paradox disappears once we review Brouwer's work from a design standpoint: *the reasoning process of the creative subject that Brouwer describes can be*

identified as a process of conception – a process by which new concepts are created by the human mind and which also is at the heart of any design process [6, 7].
In order to review Brouwer's work, from a design perspective, we first and foremost need to define what such perspective is – we need a description of design as a reasoning process. This paper adopts the description that C-K design theory provides as a framework for discussing and analyzing some of the aspects of Intuitionist Mathematics. This choice is motivated by at least two reasons. First and foremost, C-K theory is a general theory of reasoning describing design as *conceptive* reasoning process where new concepts are created and new objects are constructed on the basis of what is already known. Remark that these points are common to both Intuitionist Mathematics and to C-K theory. Second, literature about C-K design theory indicates strong relationships of the theory with the Forcing technique in Modern Set Theory. Incidentally, not only Set Theory is one of the main areas where Intuitionist Mathematics has had some of its most significant contributions, but Forcing has some notable similarities with some intuitionist notions such as free choice sequences [8-10]. This opens up the possibility to use intuitionist mathematics as a constructive foundational alternative for C-K theory and potential avenues for further research.
Let us also note, before proceeding further, that the present study can be seen as a contribution to the program announced in [11] about the necessity and potential benefits in investigating the relationship between mathematics and design. Furthermore, the parallels explained in this work have already given some formal results [12].

The plan of the paper is as follows. In Section 2, we give an overview of C-K theory and its relationship with set theory. In Section 3, we start reviewing Brouwer's work in some detail. In Section 4, we start by analyzing the Theory of Creative Subject (TCS)(a theorization of Brouwer's idealized mathematician). We argue that a major component for modeling creativity lacks to TCS – a concept space. Section 4 continues by pointing out some implicit correspondences between C-K theory and Brouwerian mathematics. Section 5 present further comments and discussions.

2 C-K DESIGN THEORY

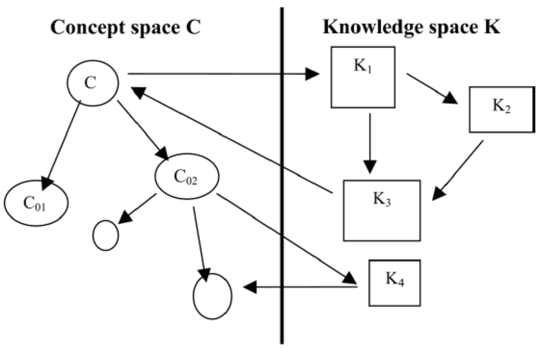

Figure 1. Concept and knowledge spaces.

2.1 Overview of C-K theory

Hatchuel and Weil [13-17] propose a theory of design reasoning which captures some of the fundamental properties of design reasoning process as a conceptive reasoning process. The theory is based on the distinction and interaction between two spaces; figure 1;

- **Knowledge space** A knowledge space represents all the knowledge available to a designer (or to a group of designers) at a given time. These are propositions that the designer is capable of declaring as true or false; i.e., propositions whose logical status are known to the designer (e.g., some phones are mobile).
- **Concept space** A concept space represents propositions whose logical status are unknown and cannot be determined with respect to a given knowledge space. These are propositions that can

be stated as neither true, nor false by the designer at the moment of their creation (e.g., some phones prevent heart attacks).

In C-K theory, concepts are descriptions of an object of the form "C: there exist an object x with the properties $p_1, p_2, ..., p_n$ such that C is *undecidable* with respect to K". In other words, the designer who created the concept is not able to tell whether such thing is possible or not at the moment of creation. A design process (or alternatively, a conceptive reasoning process begins) when such an undecidable formula is created. A designer can then elaborate the initial concept by partitioning it - that is, by adding further properties to CT there are two kinds of partitioning. *Restrictive partitions* add to a concept a usual property of the object being designed. *Expansive partitions* add to a concept novel and unprecedented properties.

Creativity and innovation are possible due to expansive partitions: such partitions lead to fundamental revision of the identity (or definition) of objects. Since concepts are elaborated by partitioning, the concept space has a tree structure.

Concepts, although different than knowledge in their logical status, they are created from knowledge. For this reason, different designers with different knowledge spaces may create different concepts. A concept space can only be defined with respect to a knowledge space – concepts are *K-relative*.

When elaborating a concept space, a designer might use his or her knowledge, either to partition further the concepts, or to attempt a validation of a given concept. This last type of operation is called K-validation and it corresponds to the evaluation of a design description using knowledge. The result of a K-validation is positive, if the designer acknowledges that the proposition "there exist an object x with properties $p_1, p_2, ..., p_n$" is true. The result is negative, if the knowledge available to the designer allows him to state that such an object cannot be built. In both case, the conception ends for the concept that has been validated (or, rejected). The reasoning may continue by creating new concepts or other (unexplored) branches of the concept tree.

Often the validation of a concept is not readily possible. Due to the expansions of the concept space, the idea of objects that the designer knows nothing about the existence has been created. In order to validate concepts, new knowledge warranting the existence conditions of such an object should be acquired. In terms of the theory knowledge should be expanded. The expansion of knowledge space is called K-expansion. The central proposition of C-K theory is thus "design is the interaction and dual expansions concepts and knowledge" [13-17].

2.2 Set theory, Forcing and C-K theory

In [15], Hatchuel and Weil attempt to bring the theory on more formal grounds. Although a complete formal treatment is not given, some of the notions of the theory are desribed using the terminology and ideas from the axiomatic set theory of Zermelo-Fraenkel (ZF). It is emphasized that the knowledge space K and the concept space C are built on different axiomatics and have different logics. In particular, if ZF is used to model the concept space, the axiom of choice cannot be used. This claim is based on the idea that during design it is not possible to choose an object having the desired properties, otherwise, this would mean that the design is already finished.

More recently, [16] presented similarities between C-K theory and Forcing. Forcing is a method used in set theory invented by Paul Cohen for the creation of new sets [18, 19]. Using Forcing, Cohen showed that, starting with a model **M** verifying the axioms of the set theory and a proposition *P* that is undecidable in **M** (i.e., it is impossible to state the truth or falsity of *P* in **M**), it is possible to construct progressively a new model **N** (containing **M**) in which *P* (or its negation) is true. According to [4], Forcing can be seen as a special case of CK design theory where set theory is taken as a *model of K space*. The proposition *P* is a (partial) description of an object whose existence cannot be confirmed or rejected in the beginning of the design process. **M** is the initial knowledge space. The conditions introduced successively correspond to properties introduced in the C space by partitioning. Design ends when a new collection of sets **N** is constructed where *P* becomes true for **N**.

3 BROUWER'S INTUITIONIST MATHEMATICS

Let us now present Brouwer's philosophy and the basic notions of Intuitionist Mathematics. We will start by describing some foundational debates that influenced Brouwer's work. Then, the first and the second *acts* of intuitionism will be presented. Some of the main ideas and notions such as the reject of the Law of the excluded middle, the Intuitionist conception of existence and the basics of Brouwerian set theory will be covered.

3.1 Pre-intuitionist debate on definability, conceivability and existence of objects

Let us start by describing the context in which Brouwer's work developed in order to better understand Brouwer's position and his contribution. Until the introduction of non-Euclidean geometry, mathematics were widely conceived as a mixture of empirical knowledge and abstractions [10]. Brouwer called this "the point of view of observation". With the introduction of mathematical entities such as spaces, species of numbers, transfinite, … this observational point of view has become obsolete, as it was difficult to find concrete counterparts of such objects. Especially, the introduction of the transfinite by Cantor (and sets of uncountable cardinalities) and the use of abstract choice principles (which never gave any individuals but classes – that may not contain discernable entities) have seen strong objections from many pioneering mathematicians at the beginning of 20th century. An intense foundational debate concerning the nature and the practice of mathematics has taken place. Russell and Hilbert have attempted to replace the previous spatial foundations of mathematics with logical and axiomatic foundations. While Russell has defended the appropriateness of *logical foundations [20]*, reconstructing, with Whitehead, the foundations of mathematics using logic [21], Hillbert conducted a massive research program to build mathematics on *axiomatic foundations*. These two approaches, reconcilable in many respects, have strong syntactic and linguistic underpinnings.

An alternative set of ideas can be found in the works of a group of prominent French mathematicians including Borel, Baire, Lebesgue, Hadamard and Poincaré. The group, called pre-intuitionists by Brouwer, has criticized thoroughly the new abstract mathematics. Poincaré expressed his doubts about the logical and axiomatic approaches of Russell and Hilbert: "The syllogism cannot reveal anything fundamentally new. If all mathematical propositions can be derived ones from others, how would mathematics not be reduced to an immense tautology?" Presuming there was more to mathematics than syllogistic reasoning, Poincaré distinguishes two kinds of minds; the analyst and the geometrician. The former uses logic and deduction while the latter uses intuition [10]. "The logic is sure, but creates nothing; the intuition is creative but fallible."

Much as Poincaré qualifies Intuition as faillible, by underlining its importance, Poincaré puts emphasis on the role of *conception* in mathematics [10]. Here lies the most consensual and essential idea that connects pre-intuitionists: what is definable is what is conceivable. Being conceivable (not necessarily by a concrete imagination) is a necessary condition for existence.

The conceivability perspective allows us to see the main objections of pre-intuitionist to Cantorian mathematics. From this perspective, the notion of *actual infinity* (infinite sets whose construction is actually terminated) becomes criticizable. Baire, Borel and Poincaré have insisted that our mind is capable only of a finite number of acts of thought. Consequently, we can only conceive finite objects and that we can consider only objects defined by a countable number of conditions – thus, thinking the actual infinity as a completed totality, not only pervades all intuition, but also is unconceivable.

3.2 First Act of Intuitionism: Mathematics as the study of mental objects

Though Brouwer agreed with pre-intuitionist on the above issues, he went beyond their critical reactions by offering an essential change in perspective and an alternative way to proceed. Beginning with his thesis in 1907, Brouwer started to build a new way of looking at mathematics. According to him, mathematics is a mental activity completely independent of the outside world, taking place in the mind of an idealized mathematician – the creative subject.

This activity takes place over the time and thus, mathematics is a constructive activity. Going completely to the opposite direction of Russell, he postulated that this mental activity, and thus, mathematics is independent and prior to all language:

> *"One cannot inquire into the foundations and nature of mathematics without delving into the question of the operations by which the mathematical activity of the mind is conducted. If one failed to take that into account, then one would be left studying only the language in which mathematics is represented rather than the essence of mathematics." Brouwer 1907, 1908 [1, 2].*

This particular view of mathematics is called the First Act of Intuitionism. In Brouwer's words, the first act

> *"[…] completely separates mathematics from mathematical language, in particular from the phenomena of language which are described by theoretical logic, and recognizes that*

intuitionist mathematics is an essentially languageless activity of the mind having its origin in the perception of a move of time, i.e. of the falling apart of a life moment into two distinct things, one of which gives way to the other, but is retained by memory. If the two-ity thus born is divested of all quality, there remains the empty form of the common substratum of all two-ities. It is this common substratum, this empty form, which is the basic intuition of mathematics. [4, 22]

Hence, the basic intuition Brouwer is proposing is based on the perception of the passage of time – the progression of a reasoning individual from one moment to the next. One of the implications concerns the nature of continuum and infinity. Since mathematics is a mental construction based on the step-by-step activity of an idealized mathematician, any notion of infinity is *potential* – not *actual*. The view of infinity as potential and the continuum as a never completed totality brings Brouwer close to pre-intuitionists who expressed doubts on the transfinite and inconceivable mathematical objects.

The emphasis on the languageless nature of mathematics puts Brouwer's conception in complete opposition with Russell's efforts of reconstructing (and thus, saving) mathematics using logic. Although, Intuitionism is not against logic (in fact, Brouwer was the first to propose a logic based on intuitionist principles), it firmly states that mathematic is *prior* to logic. *"For intuitionist mathematics every language, including the formalistic one, is only a tool for communication. It is in principle impossible to set up a system of formulas that would be equivalent to intuitionist mathematics, for the possibilities of thought cannot be reduced to a finite number of rules set up in advance. [...] For the construction of mathematics it is not necessary to set up logical laws of general validity; the laws are discovered anew in each single case for the mathematical system under consideration."* [23].

3.3 The existence of mathematical objects: an emphasis on the constructibility

According to Brouwer, for mathematical objects, "to exist" means "to be constructed" [24]. If this is not the case, then "to exist" should have a metaphysical meaning. Although Brouwer or his followers have no objection to any particular metaphysical theory, they believe that the study of mathematics cannot be related to these; *it is something much simpler, more immediate than metaphysics – the study of mental mathematical constructions* [24]. Heyting, one of the major contributors to Brouwer's program, gives the following explanation about the intuitionist view on the existence of mathematical objects [23]:

"We do not attribute an existence independent of our thought; i.e., a transcendental existence to the integers or to any other mathematical object. Even though it might be true that every thought refers to an object conceived independently of it, we can nevertheless let this remain an open question. In any event such an object need not be completely independent of human thought. Even if they should be independent of individual acts of thought, mathematical objects are by their very nature dependent on human thought. Their existence is guaranteed only insofar they can be determined by thought."

Although acknowledging that mathematical objects may be referring to the outside world, Heyting defends that those cannot be conceived independently of the human thought. The ideal mathematical objects are *accertained* to exist if they can be constructed and verified by thought.

3.4 The reject of the law of excluded middle: accepting undecided propositions

The above view on the existence of mathematical existence has some implications regarding the nature of mathematical propositions. In intuitionist mathematics, a proposition affirms the fact that a certain mathematical construction has been affected. In other terms, a proposition $P(a)$ about a mathematical object a means "there is a way of constructing an object a having a property P". This view of existence as constructibility and mathematics as the study of mental constructions accomplished over time led Brouwer to reject the idea that every mathematical assertion should be either true or false. This principle, known as the Law of the Excluded Middle (LEM) is not tenable for the kind of mathematics Brouwer proposed, since, to be able to say that $P(a)$ or its negation, $\sim P(a)$, is true for every proposition and every object a, we must have a general method for constructing any object having any property. As we have no such method for this, we have no right to use such a principle. Acknowledging such a principle would mean that laws of mathematics are general and refer to the objects of the world, independently of our knowledge. This is obviously in contradiction with the kind of mathematics Brouwer advances.[1]

One of the consequences of the reject of LEM is the possibility to consider *undecided* propositions. Since mathematics is an activity carried out over time, there may be stages where neither a property $P(a)$ of an object a, nor its negation $\sim P(a)$ has yet been decided by the creative subject.

While the reject of the LEM is only natural for intuitionist mathematics, let us note that it is a very controversial issue and it is widely held as unacceptable by many leading mathematicians such as Hilbert: *Denying a mathematician use of the principle of excluded middle is like denying an astronomer his telescope or a boxer the use of his fists. To prohibit existence statements and the principle of excluded middle is tantamount to relinquishing the science of mathematics altogether* [25].

3.5 Second act of intuitionism: generative power of intuitionism

Contrary to the first act, which is critical and destructive, the second act of intuitionism is creative and constructive [10]. Brouwer sets himself the task to deal with the continuum within the Intuitionist paradigm. However, the consideration of infinite sets as completed totalities that has been the standard mathematical practice since Cantor and Dedekind, cannot be accepted from an intuitionist standpoint. By contrast to the actual, completed infinity in the classical approach, the intuitionist approach envision infinity as *potential*. Brouwer rejects the *actual infinity* since such objects are inconceivable by a mind that disposes only of a finite number of acts of thought. In order to construe infinity as *potential infinity*, he uses the notion of *infinitely proceeding sequences* – sequences of mathematical objects that proceed to infinity but never achieve it.

In fact, pre-intuitionists have already used such sequences to define real numbers and tackle the notion of continuum (e.g. by means of classes of equivalence of Cauchy sequences for defining sets of real numbers). Yet, this was being done by sequences determined from the beginning (by a law or an algorithm) Brouwer realized that restraining oneself to such *lawlike* sequences is highly limitative: such sequences are always countable and the real numbers that can be defined by means of such sequences can only offer a countable infinity and a *reduced continuum*. Therefore, he suggests extending the classical notion of infinitely proceeding sequences to that of (*free*) *choice sequences* (or, *lawless sequences*): infinitely proceeding sequences of mathematical objects whose construction are not fixed by a predetermined law or algorithm but whose terms can be chosen arbitrarily at any stage of their construction (by a creative subject).

Allowing an act of free choice at any moment and the possibility to break away from any algorithm or law allows the consideration of *partially determined objects* and their *undecided properties*. This possibility to create new mathematical objects *at will*, provides the most eminent and original characteristic of Intuitionist Mathematics. It expands the frontier of the Intuitionist continuum from the reduced continuum to *full continuum* since the totality of mathematical objects that can be defined can no longer be determined in advance and there is always the possibility to continue defining a sequence in a way that distinguish it from all the others that are known thus far, creating thus a novel object [26].

Van Dalen [26] remarks that, compared to the classical point of view, Brouwer's universe does not get beyond ω_1 (there is no transfinite and actual infinity). But, what it lacks in 'height' is compensated in 'width' by the extra fine structure that is inherent to the intuitionist approach and its logic. This fine structure allows mathematically thinking the continuum in its very *indeterminacy* and errancy *vis-à-vis* discrete enumeration, and to do this without letting the continuum dissolve into an unintelligible mystery [9].

[1] It should be noted however that LEM holds in finite domains and the reject concerns mainly infinity**:** *"Of greater theoretical interest is the fact that LEM is also held to be valid in cases where one is operating in a strictly finite domain. The reason for this is that every construction of a bounded finite nature in a finite mathematical system can only be attempted in a finite number of ways, and each attempt can be carried through to completion, or to be continued until further progress is impossible. It follows that every assertion of possibility of a construction of a bounded finite character can be judged. So, in this exceptional case, application of the principle of the excluded third is permissible."* Heyting [12].

3.6 Spreads and Species: Intuitionist counterparts of the notion of "Set"

As Brouwer explains [4, 5], the second act *recognizes the possibility of generating new mathematical entities, firstly, in the form of infinitely proceeding sequences a_1, a_2, ..., whose terms are chosen more or less freely from mathematical entities previously acquired; in such a way that the freedom of choice existing perhaps for the first element a_1 may be subjected to a lasting restriction at some following a_ν, and again and again to sharper lasting restrictions or even abolition at further subsequent a_ν's, while all these restricting interventions, as well as the choices of the a_ν's themselves, at any stage may be made to depend on possible future mathematical experiences of the creating subject; secondly, in the form of mathematical species, i.e. properties supposable for mathematical entities previously acquired, and satisfying the condition that, if they hold for a certain mathematical entity, they also hold for all mathematical entities that have been defined to be equal to it.*

With the second act, Brouwer replaces the classical notion of set by separating it to two notions; spreads and species. A spread is a *law* that regulates the construction of infinitely proceeding sequences. The law determines

(i) which mathematical entities are accepted as the initial segment of the sequence and

(ii) which elements are allowed to further the sequence (in such a manner that there is always at least one element that should be accepted).

Figure 2 shows an example spread that corresponds to a full binary spread, i.e., infinitely proceeding sequences of 0s and 1s starting from the empty set. A spread can be narrowed down by an operation called *hemmung* (restriction, in german) to more specific spreads (e.g. the full binary spread can be narrowed down to sequences starting with < 0, 0, 1>).

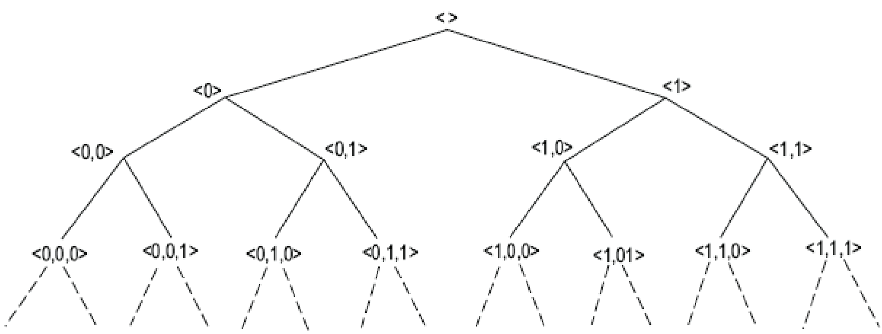

Figure 2. A graphical representation of the full binary spread.

A species is a property or a relation (or a collection of properties and/or relations). The elements of a species is precisely those that verifies the condition determined by the definition of that species – a condition for which we should have a definite way of knowing what counts as a proof of it. For instance, we can think of a species (on the full binary spread above) determined by the property P "the sum of whose first n terms is pair". For any given element (a branch of the spread corresponding to an infinite sequence), it is possible to determine whether this property holds for any given n.

The existence of a species is warranted only after a choice sequence having the corresponding property can be built (or an algorithm that will allow its construction is given). On the other hand, any property a choice sequence has can only be based on an initial segment of it, since it is not completed and there are infinitely many ways to further it.

Clearly, the notion of spread is an extensional one whereas the notion of species is intentional [8]. The introduction of species plays the role of the axiom of comprehension of classical mathematics. Spreads are on the side of 'becoming' whereas species are on the side of 'being' [10]. The possibility to generate new mathematical entities (using whatever entities that were previously acquired) with infinitely proceeding sequences and to construct new and unprecedented species provides to the subject the necessary means to study the continuum. Most of the original contributions of Intuitionist

Set Theory, such as Bar induction theorem or Fan theorem, can be interpreted only due to the introduction of ingeniously conceived lawless sequences used in the proofs. This is also true for results claiming the invalidity of LEM.

4. A C-K THEORETIC ANALYSIS OF BROUWER'S PROGRAM: INTUITIONIST MATHEMATICS AS A DESIGN PROCESS

We may now turn to discuss similarities as well as differences between the kind of reasoning processes described by C-K theory and Brouwer. We will start by discussing the Theory of Creative Subject (TCS). By contrasting this theory with C-K theory we will argue that a concept space is lacking to the model. Then, we claim that, although not presented explicitly in Brouwer's work or in TCS, Intuitionist mathematics does make use of concepts.

4.1 Theory of creative subject : a missing C space

In his expositions and lectures Brouwer considers explicitly the "knowledge state" for the creative subject. The idea of the creative subject and his knowledge state has also been discussed by others such as Kreisel, Kripke and Myhill [10]. Brouwer used this idea when giving counter-examples for classical theorems in order to show their incompatibility with the constructive nature of mathematics that he defended. Kreisel formalized principles used by Brouwer in his talks and writings describing the creative subject S and his knowledge state. The formalization is based on the passing of time in concordance with the basic intuition Brouwer proposed. The "states" are enumerated according to the succession of natural numbers. The following formalization – the theory of creative subject (TCS) - has been proposed by Kreisel. For every n and every P, either S has a proof of P, or S does not have a proof of P (for each proposition, we are always certain whether we have a proof of it or not). Having a proof of P at n is denoted $S_n \Rightarrow P$ .The following formulae try to capture the reasoning process of a creative subject:

- $(\exists n S_n \Rightarrow P) \rightarrow P$
- $(S_m \Rightarrow P \text{ and } n > m) \rightarrow S_n \Rightarrow P$
- $P \rightarrow \sim\sim\exists n S_n \Rightarrow P$
- $P \rightarrow (\exists n S_n \Rightarrow P)$

The formulae are interpreted as following: (i) If we have a proof of P (we are able to construct an object such that P holds for it) at some stage n, then P is true. (ii) If at some stage m we prove (and learn) P, then at later stages $n > m$, we still know P. (iii) If P is true, it is absurd to say that S will never have a proof of P. (iv) If P is true, then S will certainly have a proof of it in time. Note that iv is stronger than iii. Let us remark that although some controversies have arisen out of this formalization [10, 27, 28], this poses no problem for the needs of the current discussion.

What do these formulae suggest about the reasoning process of the creative subject? The first formula is a kind of rationality postulate (since it is considered that proofs are infallible). The second formula represents a principle of *accumulation of knowledge*. The fourth and fifth formulae reflect the idea that if an object with a certain property *can* be constructed, then, eventually the creative subject will get there. **Although the learning over the time aspect is included in TCS model, the "creativity" of the creative subject appears nowhere**. If every proposition P will certainly be proved at some point, then, we may ask "where is the creativity of the creative subject?"

Where do "P"s come from? - A missing C space in the Theory of Creative Subject

One of the axioms of TCS is that, if P is true, then, we will certainly have a proof of it at some stage. For a conception of mathematics where mathematics is the study of mental mathematical constructions resulting from the reasoning process of a subject, it is hardly conceivable that a subject will be able to construct every possible mathematical object and have a proof of every mathematical proposition (the only possible exception being the *idealized* subject; see previous section). If we accept that no creative subject has the time to prove all propositions that are true, then comes up the question: "which propositions should be rather proven?" Furthermore, even in the idealized case, what will be proved in one stage will condition what can be proven in the next stage. For any creative subject, there may be cases where an attempt to prove P produce an intermediary result Q which may result in the abandon of P in favor of a search for a proof of P'.

These interactive aspects between constructed objects and proofs, on the one hand, the undecided properties to be proved on the other hand is absent in Brouwer's descriptions. The Intuitionism of Brouwer provides surely an intriguing yet powerful alternative to the classical mathematics by allowing (and even welcoming) partially defined objects and considering the continuum in its full power. As we have seen, this gives the theory the possibility to *expand* the repertory of known objects and properties in new ways. However, this does not explain how the *idea* of such objects appear, neither how these ideas develop over time. Although Brouwer consider the possibility of free choice in constructing new objects, he does not recognize that such free choices go hand in hand with the choice of new *concept* of objects. It is not considered in Intuitionist Mathematics how the choice of the properties to be proved (or how these properties are chosen) may affect the next stages of the subject's activity.

Although we believe this is a lacking feature for explaining creative reasoning processes (mathematical or not), we recognize that this is only normal considering Brouwer's own motivations: Brouwer's intention was to develop and expose an alternative way of seeing the object of mathematical investigation; it was not to describe how it was being done by a subject. In fact, from his first writing onwards, what really concerned him was to defend a particular set of principles (such as the reject of the LEM), by which, he believed, mathematics should be done. The "creative subject" or the "idealized mathematician" were merely notions he used for illustrating his convictions and explaining his vision. Although he used notions like "creative" or "creating", Brouwer never set out to explain (mathematical) creativity itself.

By contrast, it is the very *raison d'être* of C-K theory to capture how new objects can be conceived and constructed *creatively* and the theory claims that there *is* a concept space at work whenever there is such a creative endeavor. By contrasting to Brouwer's creative subject, we can better understand why a concept space is important: *in the context of creativity, learning does not happen randomly and without purpose*; there is an intention behind every creative act. C-K theory captures this intentionality *and* its dynamic nature with the C space – a feature that lacks to the Theory of Creative Subject.

Nevertheless, despite this apparent difference in Theory of Creative Subject and C-K framework, the constructive reasoning process Brouwer describes can still be cast within the C-K theory. We shall discuss this next.

Notions of C-K theory	**Brouwer's ideas and notions of Intuitionism**
Designer	The mathematician; the creative subject, the constructor
Knowledge	- All the objects, in particular sequences, and species that has been acquired so far. Those include lawlike sequences. - All the (proven) properties of the objects constructed so far.
Concepts	Species that are yet unproved to exist; propositions that are yet undecided
Objects	Infinite proceeding sequences, spreads, species (and other conventional objects such as the natural and rational numbers)
Object whose design are finished	Finite objects or Lawlike sequences
Object that accept "expansions"	Lawless sequences
Expansions	Properties or objects obtained by "Free" choices
Restrictions	Properties or objects obtained by Hemmung
Operators / Interactions of the C and K	-

Table 1. Correspondences between notions of C-K theory and Brouwer's work

4.2 "Creative subject" as a designer: Correspondences between C-K theory and Intuitionist Mathematics

The reader may have noticed during the presentation that there are numerous similarities between Brouwer's approach to the mathematics and some of the main ideas suggested in C-K design theory. Some of these correspondences are recapitulated in table 1. Let us try to summarize similarities and connections by discussing Brouwer's ideas from the viewpoint of C-K theory.

The creative subject as a designer

Perhaps the most controversial trait of Brouwer's mathematical philosophy is his constant and explicit reference to the mathematician in order to argue and justify the principles by which mathematics should be done. This is quite different, for instance, from the approach of Hilbert who proposed the axiomatic method as a general method for the mathematics. By contrast to him and still others, Brouwer speaks about a subject, the mental constructions of this subject and his or her freedom to choose about how the activity should proceed. This is not to suggest that truth is relative, but to address the question "how mathematics should be done" by pointing out "who does the mathematics".
Let us note that, although he does not mention the creative subject until his late writings [3], the place he gives to the intuition of the subject is apparent even from his early work [1, 2]. In fact, as Niekus [27, 28] remarks, he uses similar but different terms for the mathematician – creative subject, creating subject or even constructor. No doubt, in Brouwer's view mathematical reasoning is an activity of creation and construction. The subject creates new mathematical objects over time and the objects that have been created can be *reused* to create further objects. These characteristics already bear some significant similarities to the way a designer's reasoning process is captured by C-K theory.

- The Process Brouwer describes is a creative and constructive process,
- New objects and properties appear over time; they become available to the subject who learns them,
- What can be created next depends on what the subject knows at this stage.

The first point is the *sine aque non* of a design process. The second point is a natural consequence of the process that is captured by the idea of K-expansion in C-K theory. The last point can be just considered another way to state the K-relativity principle in C-K theory. The "knowledge space" of the creative subject can be identified as the objects he or she created thus far and the properties of these objects he or she knows of (see table 1).
Although, Brouwer does not explicitly suggest a single notion that corresponds to concepts, they are indeed very much present in Intuitionist Mathematics: the undecided propositions. As we have seen, the primary effect of rejecting LEM is to allow undecided propositions. By separating the notion of set to its intensional and extensional components (i.e. species and spreads), Brouwer approaches even closer to capturing the essence of concepts. While spreads provide a general means to construct new objects extensionally, species whose existence are not assertained at the current stage conduct and orient this construction. It seems natural to draw a correspondence between the interplay of spreads and species, on the one hand, the interplay of concepts and knowledge, on the other. Remark, however, that there is no representation or even mention of such an interplay. Nonetheless, there is nothing that *prohibits* it either. A creative subject may very well construct (or at least, attempt to construct) new objects by means of spreads or free choice sequences in order to prove the existence of some species. Reciprocally, he or she can project some undecided species for which to look for a proof. Admittedly, this is the same kind of interaction between intensional object definitions in the C space and extensional objects in the K space that C-K theory describes.
Finally, we can observe some parallel between the notions of expansion and restriction of C-K and the means of object construction in Intuitionist Mathematics: narrowing down already defined spreads by means of Hemmung and modifying choice sequences by free choices can be seen as vehicles of obtaining restrictive partitions and novel properties in C space.

5. SOME FUTURE PERSPECTIVES

Choice sequences and Generic sets: Intuitionist mathematics as a model of CK

Intuitionist free choice sequences are similar to the notion of Generic Sets used in Forcing [8-10]. In Forcing, it is by means of the Generic Set (and its combination with the sets of the old model by a technique called *naming*) that the new model is constructed. The generic set, itself, is a construction built by using the sets of the old model but a construction which is not contained in the old universe. Compared to the generic sets of Forcing, intuitionist free choice sequences can be called *potentially generic* [8-10]. From the intuitionist point of view, it is not ever possible to reach an actual infinity like the generic sets (its construction will never be completed). However, free choices allow, *in principle*, to build a set (i.e. construct a sequence) which will be different, at any moment, from every other set constructed thus far. This issue provides yet another interesting connection between Intuitionist mathematics and C-K theory. This will be addressed in future work.

Choice, decision and design

The *free choice* Brouwer introduces emphasizes an important aspect in the creative process: the creative subject can choose how to construct an object. By contrast to Cohen's Forcing method that emphasizes "finding" appropriate conditions for creating new worlds (models that contain unprecedented objects), Brouwer's choice sequences emphasize "Choice" of properties to be bestowed upon constructions. The connection we establish between C-K design theory and Intuitionist Mathematics gives us a way to reintroduce the important notions of choice and decision in design, but in a way completely different than the classical conception of these terms: in the traditional decision making and choice settings, the objects of decision (alternatives to be evaluated) are completely (and often, extentionally) defined objects. Consequently, all their properties are fixed and *knowable* even if they are not currently known or there is uncertainty about them. In design context, however, objects are partially defined and all their properties are not known. It is possible that the theoretically solid, rigorous formal grounds offered by Intuitionist mathematics about partially determined objects may help to tackle this issue.

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