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ANALYSIS OF 90° POP-UP STRUCTURES FOR CAD SYSTEMS

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Abstract

Traditionally, designs of crafts like paper pop-up structures are manually prepared and typified by excessive trial-and-error work. In recent years, attempts have been made to enable more efficient crafting methods on computer-aided platforms. This paper discusses properties of paper engineered, 90° pop-up structures and how they could be applied to CAD systems for the design of such crafts. In particular, graph theory was used to investigate their structural properties. Planar graphs are found to be useful in representing two-dimensional drawings of pop-up structures on CAD systems. The graphs enable design verifications using Grinberg's Theorem and topological conditions of crease and cut edges. A set of constraint-based crease equations leading to a boundary region further verifies creases' constructions.

Keywords: Paper engineering, computer aided design, pop-up structures, geometry, topology

1. Introduction

Paper engineering has been more valued for its artistic achievements in paper crafts than being recognized for its technological potentials. The crafts' construction techniques like folding and cutting are often deemed a form of low technology [1] and seems irrelevant to modern digital advancements. The emerging field of origami science, however, has offered a new ground for study in the area of physics, engineering, genetics and computing. In particular, origami mathematics has unveiled fresh comprehension in mathematics by paper folding [2], [3], [4]. Similarly, pop-up structures are found to exhibit useful geometric properties [5], [6], [7]. With these findings, the digital world and the traditional crafting techniques have a channel to integrate, leading to the birth of a family of computational crafting tools. For both the individual craft enthusiast and the printing industry, such tools would desirably reduce trial-and-error tasks, thereby saving materials and time in production. Paper crafts like origami and paper models have already been experimented with robust design softwares such as Treemaker [8] and Pepakura Designer [9].

Attempts have also been made to enable design of pop-up structures on computers. Notably, Jun Mitani's 3D Card Maker [10] and Sue Hendrix's work on Pop-up Workshop [11] are applications that promote pop-up constructions using computer-aided design. However, the unique craft of pop-up structures is characterized by reversible transformations from two-dimensional to three-dimensional space and a broad variety of linkage mechanisms. Existing applications are limited in the understanding of the craft's structural attributes, and consequently, they do not facilitate the design of all types of pop-up structures. Hence, to realize a generalized computer-aided system with a systematic approach for pop-up designs, an extensive study in topology and geometry of the craft is vital.

The following section of this paper introduces the basics of pop-up structures. Section Three analyzes 90° pop-up structures using graph theory, particularly highlighting Hamiltonian plane graph and the application of Grinberg's Theorem. Section Four examines the properties of crease and cut edges. Section Five describes how the graphs and their properties could be used on CAD systems. The paper concludes in Section Six.

2. Elements of pop-up structures

Pop-up structures can generally be grouped into two types, namely the 90° and the 180° structures [12], as shown in Figure 1. The angle refers to the angle between the two base pages, at which the pop-up structures fully erect. Pop-up books and cards may also contain fully erected structures at 270° or 360° [13]. But they are in actuality the combinations of 90° and 180° structures. 90° pop-up structures can be made with multi paper pieces jointed by gluing or constructed from a single paper piece with slits.

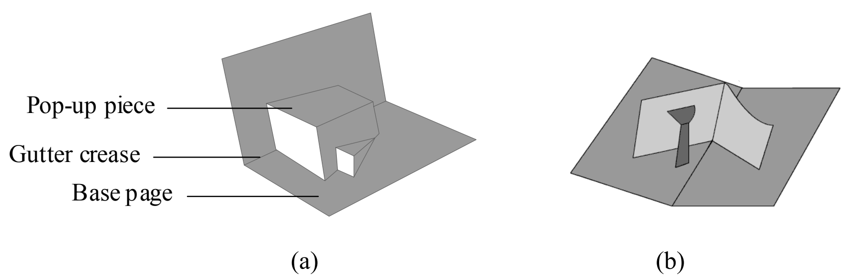


Figure 1. (a) A 90° pop-up structure and (b) a 180° pop-up structure

An essential feature of these movable pop-up structures is their ability to fold flat during a transformation from 3D to 2D space. The structure is flat foldable when the pop-up pieces are able to collapse between the two base pages as they are closed together. In this paper, the focus of investigation is the one-pieced 90° pop-up structures and pop-up pieces are assumed to be always planar. Moving paper mechanisms like the wheel and the pull-tab, which do not lead to 3D transformations, are excluded from the study.

A 90° pop-up structure comprises two base pages and pop-up pieces in layers. Each layer is made up of two pop-up pieces and additional folds may be added on it. The structure could be further broken down by a representation of vertices, edges and faces, as illustrated in Figure 2. The definitions of these elements were given as follows.

2.1 Edges

An edge is a side of a pop-up piece or a base page. There are two types of edges, namely the crease edge and the cut edge. If a line is produced by folding, it is termed as a crease edge. The gutter crease, a fold that separates the two base pages, is such an edge. If it is created by a cut or slit on the paper, it is termed as a cut edge. Graphically, crease edges are represented by solid lines and cut edges by dash lines in the following sections of this paper.

2.2 Vertices

A vertex is a point where two edges meet. As with the edges, there are two distinct vertex types. A point of intersection of solely crease edges is termed a crease vertex, and a point of intersection of cut and crease edges is termed a cut vertex. The former is located within a paper piece and leads to a flat vertex fold [14], [15]. The latter is located on the edge of a paper piece and results in a pop-up transformation.

2.3 Faces

A face refers to the area bounded by crease and cut edges. Likewise, there are two types of faces. A solid face is bounded by a combination of crease edges and cut edges. It represents a pop-up piece or a base page. A cut face is bounded by only cut edges. It is a hollow area and can represent a slit on the paper or a cutout. In addition, there exists an exterior face on any graph. It is referred to as the unbounded face and would be applied in Hamiltonian graphs in Section 3.3.

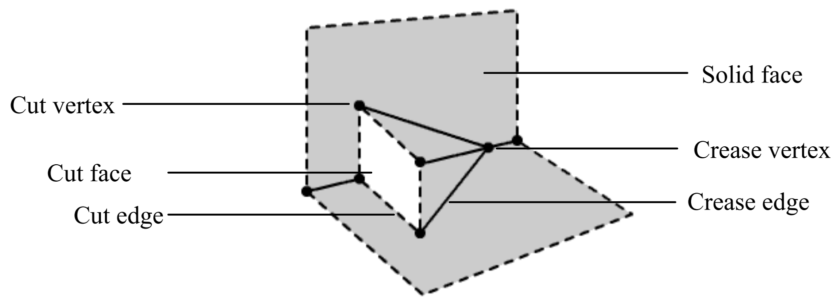


Figure 2. Edges, vertices and faces on a 90° pop-up structure

2.4 Boundary consideration for base pages

A piece of rectangular paper is usually used in the construction of a pair of base pages for a 90° pop-up structure, as shown in Figure 3a. It is also feasible to develop the structure from a piece of paper of other desirable shapes, consisting of a number of edges other than the usual four. See Figure 3b. This raises an issue on how the edges and vertices could be examined if there were so many variations to the shape of the base pages.

As the main interest in this study is the cut and crease edges that led to the development of the structures, the number of edges and vertices on the base pages would not affect the topological attributes of the structures. They are, therefore, of trivial importance here. As such, let us consider a boundary line to define the faces of the base pages. This is represented by one cut edge on each base page, as illustrated in Figure 3c.

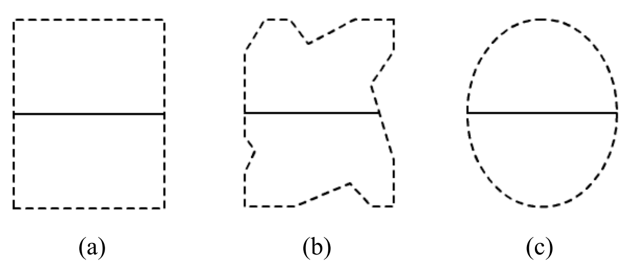


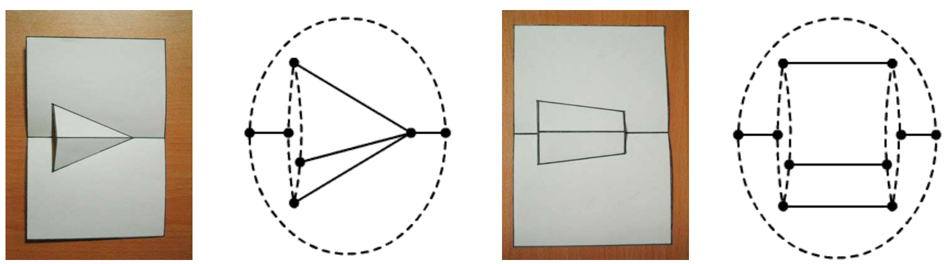
Figure 3. Boundary line for base pages

Likewise, the sides of the pop-up pieces can be cut to form many variations but the structural compositions of the pop-up designs would not change. Hence, only the elementary pop-up types, the single-slit and the double-slit, are examined in the following sections.

3. Graph forms of 90° pop-up structures

If the faces and edges of a 90° pop-up structure were to be modeled as a graph, it would form a planar graph [16]. A graph is planar if there exists a drawing of it in the plane in which no two edges intersect in a point other than a vertex. This means that the edges do not cross one another, regardless of the structure's state, be it flat folded or erected. If edges cross, it could indicate that the jointed paper pieces are twisted or interlocks exist between them, and the structure would consequently not be flat foldable. Hence, the planarity of a graph corresponds to a structure's ability to flat fold. However, this is an essential but not a sufficient condition as geometry on the structure further affects flat folds.

Let us denote a graph of the pop-up structure as *G*. On *G*, a single-slit is characterized by a cut face, a crease vertex and a cut vertex. For a double-slit, *G* has two cut faces and two cut vertices. Flat vertex folds can further be added on any pop-up piece and result in additional crease vertices on *G*.

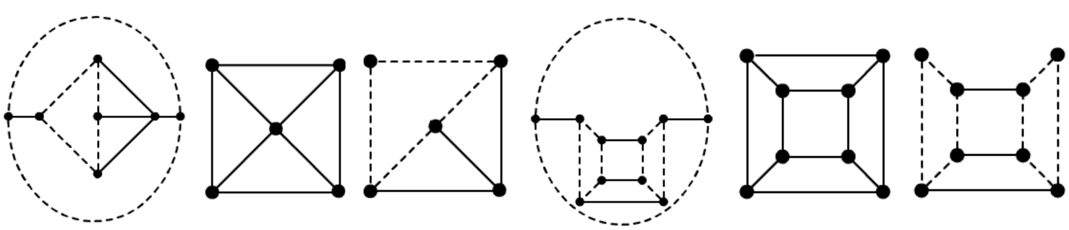


(a) (b) (c) (d)

Figure 4. Graphs of pop-up structures. (a) The single-slit and (b) its graph, and (c) the double-slit and (d) its graph

3.1 Graphs of the single-slit and the double-slit

The topology of a pop-up structure does not change when it is transformed from 2D to 3D space and vice-versa. The graphs in Figure 4 illustrate views that are typically produced by 2D drawings. Similarly, they can be displayed in isomorphic forms to better represent the structures in three-dimensional views.



(a) (b) (c) (d) (e) (f)

Figure 5. (a) The single-slit graph, (b) a pyramid graph, (c) a sub-graph of the single-slit, (d) the double-slit graph, (e) a cube graph and (f) a sub-graph of the double-slit

Figure 5a shows a graph of the erected single-slit, which is isomorphic to that in Figure 4b. Notice in Figures 5a and 5b that the graph of the single-slit and the polyhedral graph of the semi-octahedron or pyramid contain a common sub-graph. However, a single-slit does not always have a pyramid-like structure. Figure 6c illustrates a variation. Similarly, Figure 5d is a graph of the double-slit, isomorphic to that of Figure 4d. It has a common sub-graph (Figure 5f) as the cube graph (Figure 5e). Figure 6 illustrates the elementary structures in their erected forms and their graphs.

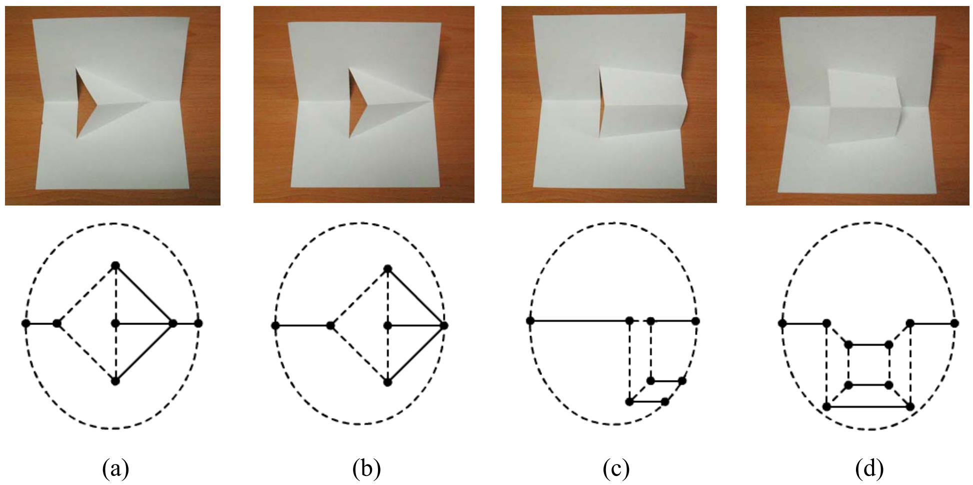


Figure 6. Erected 90° pop-up structures (top) and their corresponding graphs (bottom)

3.2 Base graphs and layers

The two sub-graphs in Figures 5c and 5f form the building units of pop-up layers. Let us term them as the base graphs. Additional pop-up layers can be represented by attaching these base graphs onto the crease edges of an existing graph. This is achieved by dividing a crease edge of the existing graph into two and joining their ends with vertices of the base graphs. But not all vertices of a base graph can be jointed with the crease edge. Section 4.1 explains the topological conditions essential for the merger of graphs.

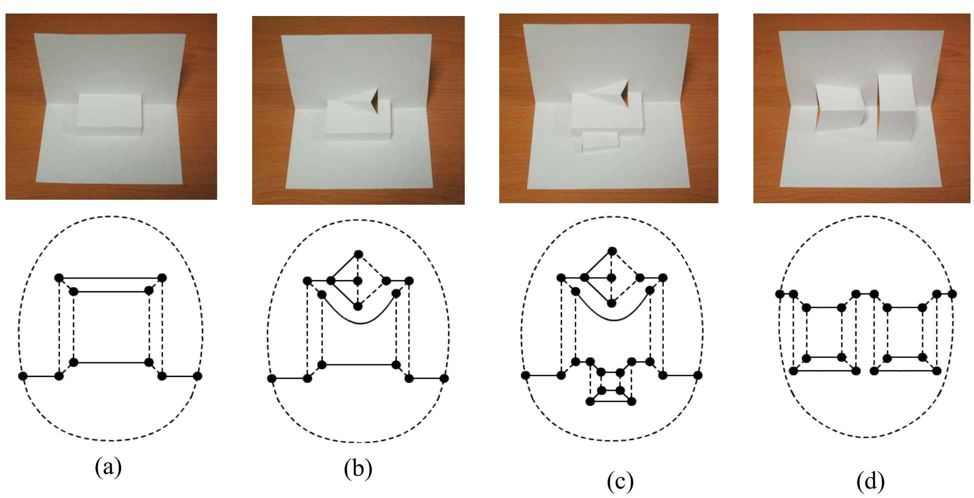
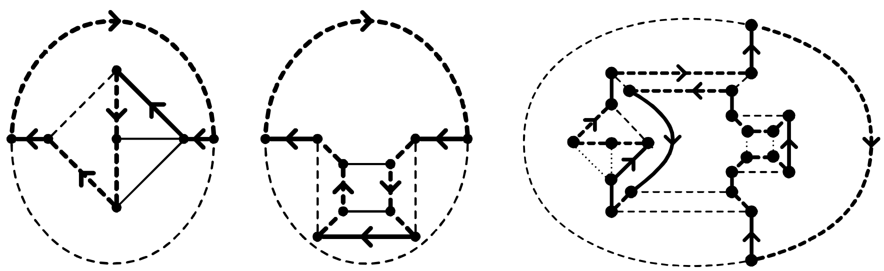


Figure 7. The graphs depicted the development of a 90° pop-up structure from (a) the first layer, to (b) the second layer and finally (c) the third layer. (d) A graph of a structure made up of two double-slits.

Figures 7a to 7c give an example of a graph development for a three-layer pop-up structure that was made up of two double-slits and one single-slit. Figure 7d shows a graph of two-layer pop-up structure comprising double-slits.

3.3 Hamiltonian plane graphs

If the planar graph is further embedded on a plane, it is a plane graph. *G* is observably a plane graph since an one-piece pop-up structure is made from a planar piece of paper. On *G*, a closed trail passes through all vertices. Such a trail is a Hamiltonian cycle and a characteristic of plane graphs. Figure 8 shows possible trails (bold lines) for Hamiltonian cycles on some graphs.



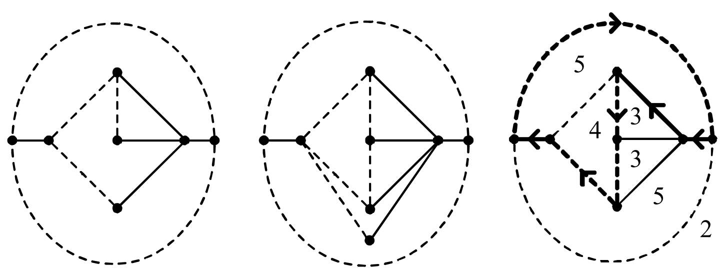
(a) (b) (c)

Figure 8. Hamiltonian cycles of (a) a single-slit, (b) a double-slit and (c) a three-layered pop-up structure.

To verify that *G* is Hamiltonian, Grinberg's Theorem [17, 18] could be applied. The theorem states that if a loopless plane graph has a Hamilton cycle *C*, then

$$\sum_{i=2}^{n}(i-2)\left(\phi_i' - \phi_i''\right) = 0 \qquad (1)$$

where ϕ_i' and ϕ_i'' are the number of faces of degree *i* contained in the interior of *C* and exterior of *C* respectively. If the theorem is not satisfied, G is not Hamiltonian and may contain vertices and edges leading to a flawed design of a pop-up structure. For example, Figure 9a illustrates a non-Hamiltonian graph that cannot model a pop-up structure. The non-Hamiltonian graph in Figure 9b has a crease edge and a cut edge that are redundant.



(a) (b) (c)

Figure 9. (a) and (b) are non-Hamiltonian graphs. (c) The numbers indicate the degree of the faces on the graph.

To exemplify the application of the theorem, the Hamiltonian cycle on the graph in Figure 9c was analyzed. The degree of a face is the number of edges that bounds the face. In the interior of the cycle, there are two faces, one of degree 4 and 5. On the exterior of the cycle, there are four faces, one of degree 2 (unbounded face), two of degree 3 and one of degree 5.

$$\begin{aligned}&\sum_{i=2}^{n}(i-2)\left(\phi_i^{'}-\phi_i^{''}\right)\\&=(2-2)(0-1)+(3-2)(0-2)+(4-2)(1-0)+(5-2)(1-1)\\&=0\end{aligned}$$

Thus Grinberg's Theorem offers a method to validate the graph structure. However, this validation is not adequate, as it cannot inspect the nature of the edges or the localized configurations of the crease and cut edges. In the next section, relationships of crease edges and cut edges will be analyzed. This includes an examination on the topological conditions for cut and crease edges and a previous study [19] on the boundary region for creases.

4. Properties of crease and cut edges

4.1 Topological conditions for edges

Several conditions govern the topology of crease and cut edges. Technically, they define how the folding and cutting techniques can be appropriately applied together to develop the pop-up structure. On the graphs, these conditions ensure that the topological elements are correctly connected together, inclusive of developments in additional flat vertex folds and successive pop-up layers on a structure. The cut edges in the following conditions belong to slits that would lead to pop-up formation. They do not refer to those of cutouts.

1. The number of adjacent cut edges on G is always two.

   In order for a pop-up layer to fold, crease edges have to intersect cut edges. A slit across a crease edge creates a pair of cut edges jointed at the intersection, which is also the cut vertex. Additional adjacent cut edges at the vertex do not produce pop-up layers and is redundant. It is like producing a tear on the paper.

2. The two cut edges should, individually or together, be part of a cycle of cut edges of at least four, and the sum of the cut edges in the cycle is always an even number.

   This is essential for the formation of a cut face on the pop-up structure, which is bounded by at least four cut edges. Furthermore, cut edges are the sides of solid faces. It has been shown that the number of solid faces is always even [19]. Hence, since the number of solid faces is even, the number of cut edges must also be even.

3. At least one crease edge has to be adjacent to the pair of cut edges on G.

   Related to condition 1, there must be crease edges adjacent to the cut edges so that pop-up layers can be developed and raised. There is no upper limit to the number of crease edges adjacent to the cut edges.

4. The number of crease edges not adjacent to any cut edge is even.

   These crease edges intersect at the crease vertex, leading to single vertex folds. The condition can be derived from Maekawa's theorem [14].

4.2 Boundary region for creases

There are two types of creases, namely mountain creases and valley creases. A crease can be a mountain crease or valley crease, depending on the side the paper is facing up. See Figure 10. In the previous study [19], the combination of these crease types to build a 90° pop-up structure was examined. The crease types are found to exhibit sequential patterns as pop-up layers are added. The patterns are expressed in equations and set on a graph termed *MV* graph. In turn, the lines describing the equations form a boundary region, as illustrated on Figure 11. The region encloses the domain for feasible constructions of the structures and their abilities to flat fold. These findings do not use graph theory and do not entail cut edges.

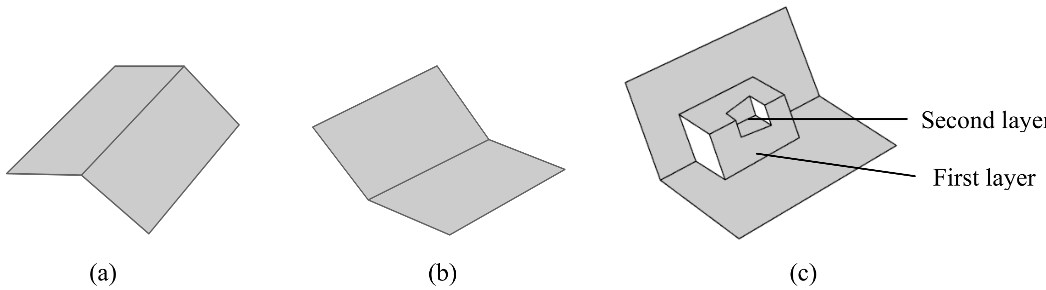


Figure 10. (a) A mountain crease, (b) a valley crease and (c) a two-layered structure

In the equations, *M* and *V* denote the number of mountain and valley creases respectively. The gutter crease is taken to be a valley crease unless otherwise stated. *M* and *V* would interchange if the gutter crease is assigned as a mountain crease. To form the first pop-up layer, two additional valley creases and one mountain crease are required. Therefore, to create a 90° pop-up structure, there must be at least four creases. This gives a boundary inequality

$$M + V \geq 4. \tag{2}$$

There are two ways to add layers to a structure. For example in Figure 10c, the first layer of the structure was built over a valley (gutter) crease and the second layer was built over a mountain crease. When successive pop-up layers are built on existing mountain creases on the structure, the relationship between mountain and valley creases can be expressed as

$$M = 2V - 5,\ V \geq 3. \tag{3}$$

On the other hand, if the successive layers are constructed on existing valley creases, the relationship becomes

$$V = 2M + 1,\ M \geq 0. \tag{4}$$

The latter two equations bound the relationships of crease types as increasing number of layers are built, and the boundary region formed is semi-infinite. Though there is no theoretical upper limit to the creation of layers and creases, it is known that physical properties, such as the thickness of the paper, limit the number of layers to be added.

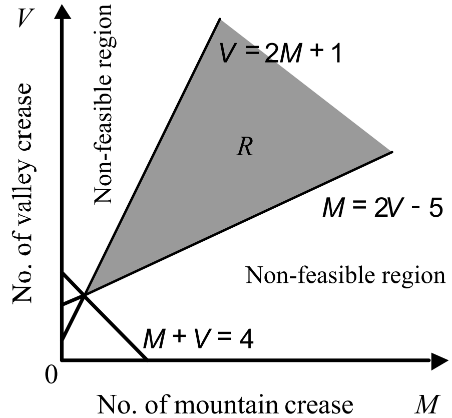


Figure 11. The feasible region R for creases' constructions on 90° pop-up structures

On the *MV* graph, it is also feasible to trace a path within the region. The path depicts the development of creases and layers. In Figure 12, the solid lines represent pop-up layers. The dotted lines represent the creation of new layers over mountain creases. Mathematically, each is characterized by a gradient of 2. On the other hand, the dash lines represent the creation of new layers over valley creases and each has a gradient of $\frac{1}{2}$. The intersections of these lines locate possible combinations for the numbers of mountain and valley creases.

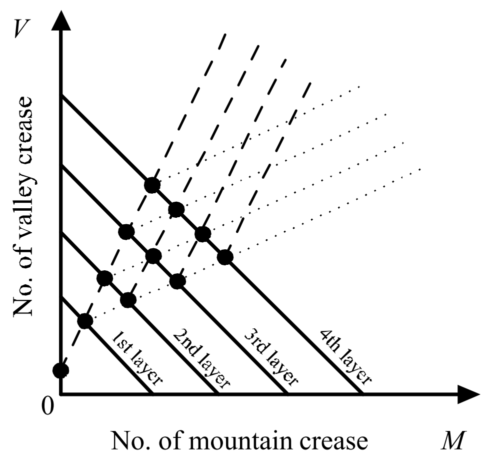
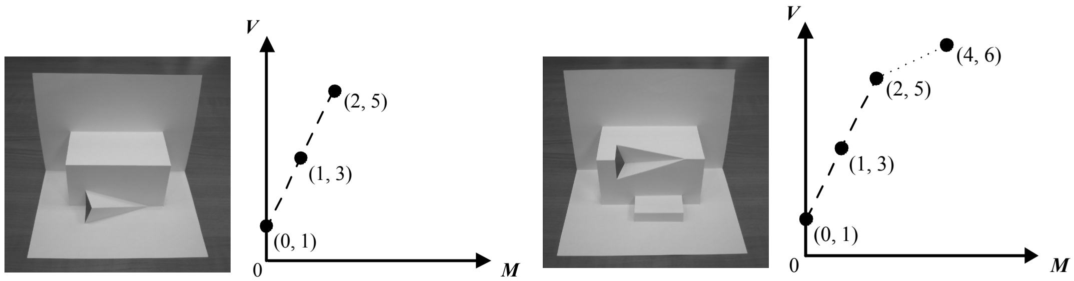


Figure 12. Paths depicting development of creases and layers

Figure 13 shows two examples of *MV* graphs representing the crease characteristics of corresponding pop-up structures. The point (0, 1) represents the gutter crease, which is a valley crease. In Figures 13a, the first layer is achieved by creating a double-slit over the gutter crease. It is represented by the point (1, 3) in Figure 13b. The second layer is a single-slit added over a valley crease on the first layer. By the second layer, the structure has a total of two mountain creases and five valley creases and is represented by (2, 5) on its *MV* graph. In Figure 13c, the third and last layer is created over a mountain crease, as shown by the dotted line on its graph (Figure 13d). As shown on that graph, a total of four mountain creases and six valley creases embody the structure.



(a) (b) (c) (d)

Figure 13. MV graphs. (a) A two-layered structure and (b) its corresponding graph, and (c) a three-layered structure and (d) its graph

Thus, the identification of the boundary region enables the viable permutation of mountain and valley creases that resulted in foldable pop-up structures. Furthermore, the development of the structures could be shown on *MV* graphs. These findings would further complement the study of crease edges in graph theory when the assignments of crease types are to be specified.

5. Applications on CAD systems

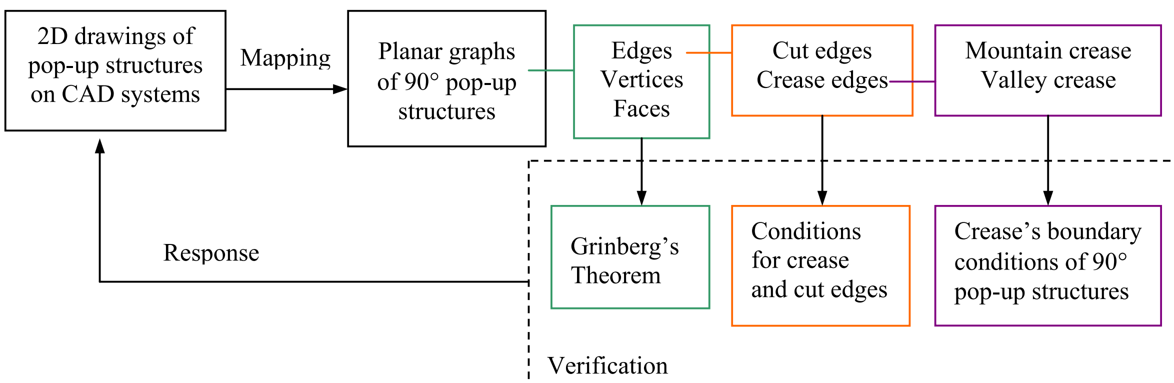


Figure 14. A possible structure for CAD systems

By mapping 2D outlined sketches of pop-up structures on paper as planar graphs, the design of the structures can be digitally realized and presented on CAD systems. The design can be represented as topological elements and verified in three levels, as shown in Figure 14.

- The first level validates a feasibly drawn design by examining if all vertices can be connected in a single cycle of edges. This can be mathematically performed with Grinberg's Theorem, which involves calculation on faces, when applied on CAD systems.
- The second level looks into the type of edges making up a pop-up structure and matched with topological conditions of crease and cut edges surrounding a face or intersecting at a vertex.
- The third level analyzes the assignments of crease types and their combinations on a structure. Any combination of mountain and valley creases that falls out of the boundary region would render the design inapplicable.

These verifications are helpful in minimizing manual trial-and-error work such as the check to ensure the structures' flat foldability and sufficiency in number of creases. On CAD systems, they would form part of the backbone that supports a set of functions on the user interface, which permits the building of pop-up structures. The systems would also contain another set of functions that enables artistic capabilities like colouring and the insertion of texts and graphics. Besides the utility to print on paper, they would additionally incorporate the rendering of three-dimensional views and animations of the pop-up mechanisms when opening and closing the base pages.

6. Conclusions

90° pop-up structures can be represented by planar graphs. Properties of these graphs can be applied to 2D drawing functions of pop-up structures in computer-aided systems. In addition, the crease and cut edges of these structures hold topological relationships and conditions that are useful for design verifications, especially in the structures' ability to flat fold. However, topological correctness does not mean that a structure is geometrically possible. There are cases where topological conditions are satisfied but flat folds cannot be achieved due to interferences of adjacent moving pop-up layers. For example, no pairs of crease edges adjacent to a pair of cut edges should form an angle less than 180° on the base pages. If this geometric condition were violated, the structure would not be flat foldable. Therefore, geometric limitations in the structures' movements require a more thorough study so as to effect a deeper understanding in the craft's design.

An extension from this study has also led to the creation of a methodology on how a polyhedral can merge with a pair of base pages and become flat foldable. This is described with properties of crease and cut edges. In order to realize a generalized CAD platform, our on-going work has also encompassed the study of intrinsic characteristics exhibited by 180° pop-up structures. The study is further supported by a CAD prototype, presently in the process of construction. This project would ultimately benefit paper engineers and craft enthusiasts, and bring about enrichment to the design environment of pop-up structures, one that could appropriately aid the learning of the craft in this digital age.

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